The James Webb Telescope

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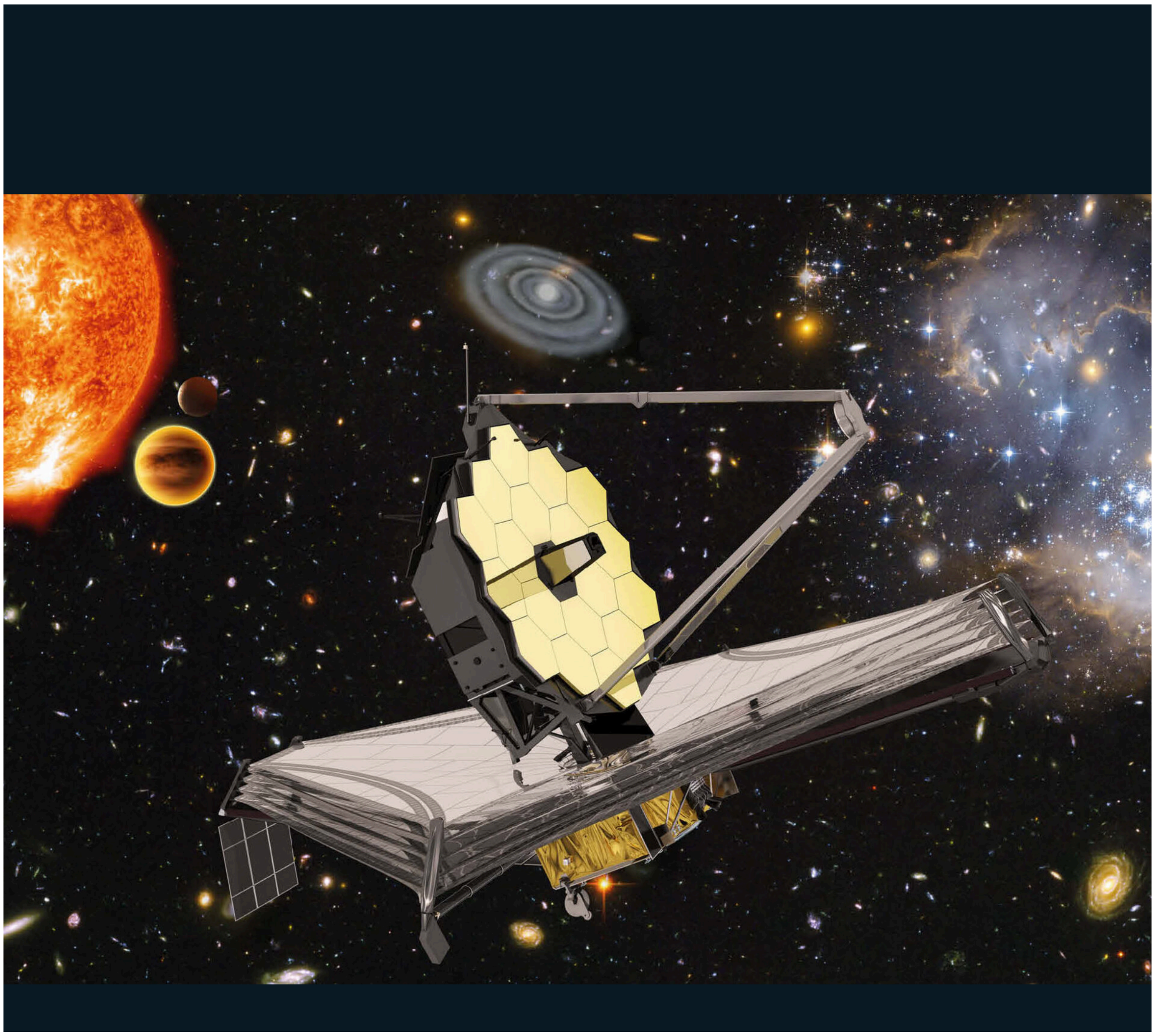


Artist's Conception of the James Webb Telescope

These notes began with my interest in exploring the orbital mechanics of the James Webb Telescope. My research, however, uncovered a number of interesting facts regarding the project including: the complexity of the project, the energy-saving positioning of the instrument, and the strange oscillations of the planned orbit.

Complexity of the Mission

The James Webb Space Telescope (jointly developed by NASA, the European Space Agency, and the Canadian Space Agency was launched on 25 Dec 2021. As lead industry team tasked with building the instrument, Northrop Grumman has posted video describing the mission. The telescope is designed to peer back in time to an era near the "Big Bang" when galaxies were forming. It is an infrared telescope designed to analyze this band of the electromagnetic spectrum. It needs to be very cold and in a stable position relative to earth to accomplish its mission. In viewing the video, one comes away with an understanding of the complexity of the mission. From a strictly mechanical point of view, deploying this instrument requires flawless operation of hundreds of complex steps. Thermal management is essential to proper functioning. On the sun-side of the instrument temperatures approach the boiling point of water, while on the deep space pointing direction the instrument must be kept at -380° F (just 45° above absolute zero).

Positioning the Webb

In 1772, Italian-born astronomer Joseph Louis Lagrange discovered five points in space that an object will remain fixed relative to the earth and the sun. These points are called Lagrangian points; they were discovered as a consequence of studying the t*hree body problem*—earth/sun/object[1].

While the three body problem is very difficult to solve analytically, we can look at the plane of the earth's orbit and map contours of potential in which the total gravitational force[2] is represented by gradientin the earth/sun system. The more densely packed the lines, the greater the gradient of the force. From the diagram on the next page one can visualize this field. The Lagrangian points are labeled L1, L2, L3, L4 and L5. These are *equilibrium points* where the *net force (including the centripetal force[3])* acting on an object by the sun and the earth is zero. A satellite placed at

[1] I'd write satellite here, but in 1772 there was no notion of an artificial satellite.

[2] gravitational forces exerted by sun and earth as well as centripetal force as a consequence of the motion of the object.

[3] The centripetal force arises from the acceleration toward the center of gravity for an orbiting object. In Newtonian physics, it is the force that keeps the object in orbit.

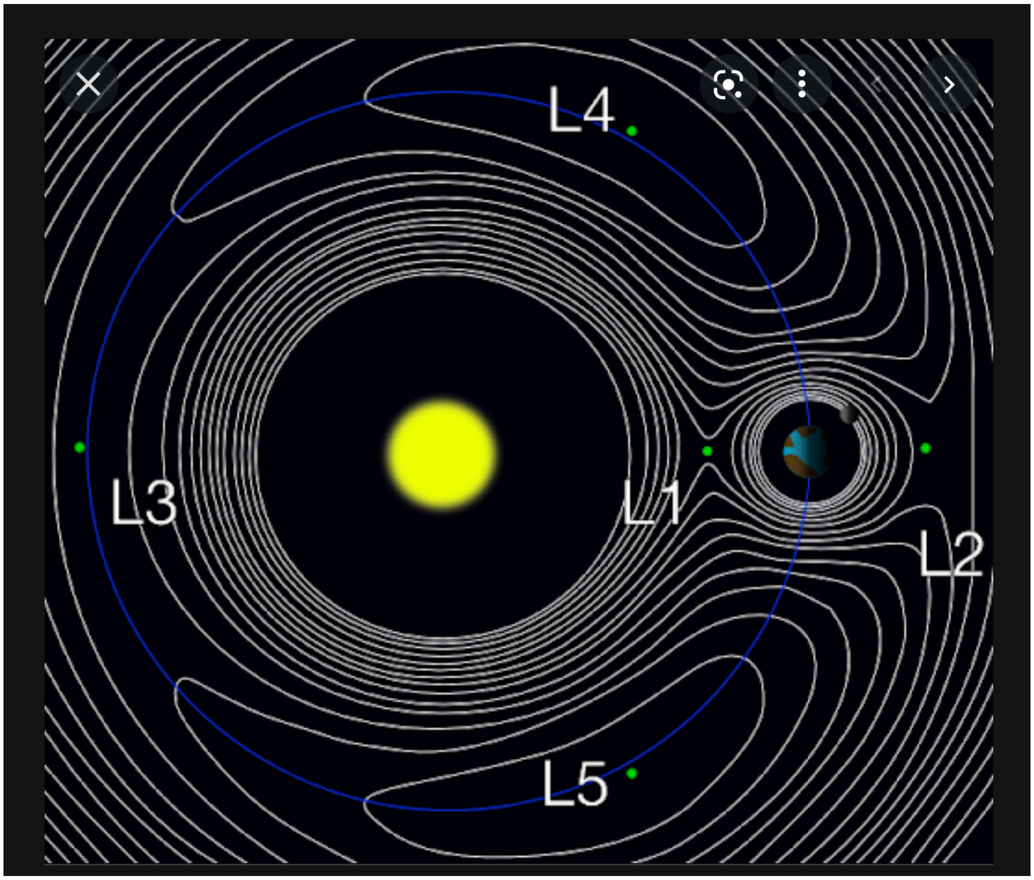


Fig 1: Gravity wells Earth/sun system

these points will remain at that position, requiring very little fuel consumption to maintain its position.

Clearly, L2 is a likely choice from which to explore the deep recesses of the universe. The other points are too close to the sun creating problems in maintaining temperature control for the infrared sensing cameras.

As the earth orbits the sun, so do the Lagrangian points. You can picture this by understanding that the graphic shown in Fig. 1 is viewed from above with a camera which rotates as the earth orbits the sun. If this is the case, each of these Lagrangian points will orbit the sun at the same angular velocity as does the earth. That is, the gravitational forces associated with the earth/sun system rotate with earth's orbital period, and *any satellite placed at a Lagrangian point will orbit the sun at earth's angular velocity*.

The Position of L2: The Restricted Three Body Problem

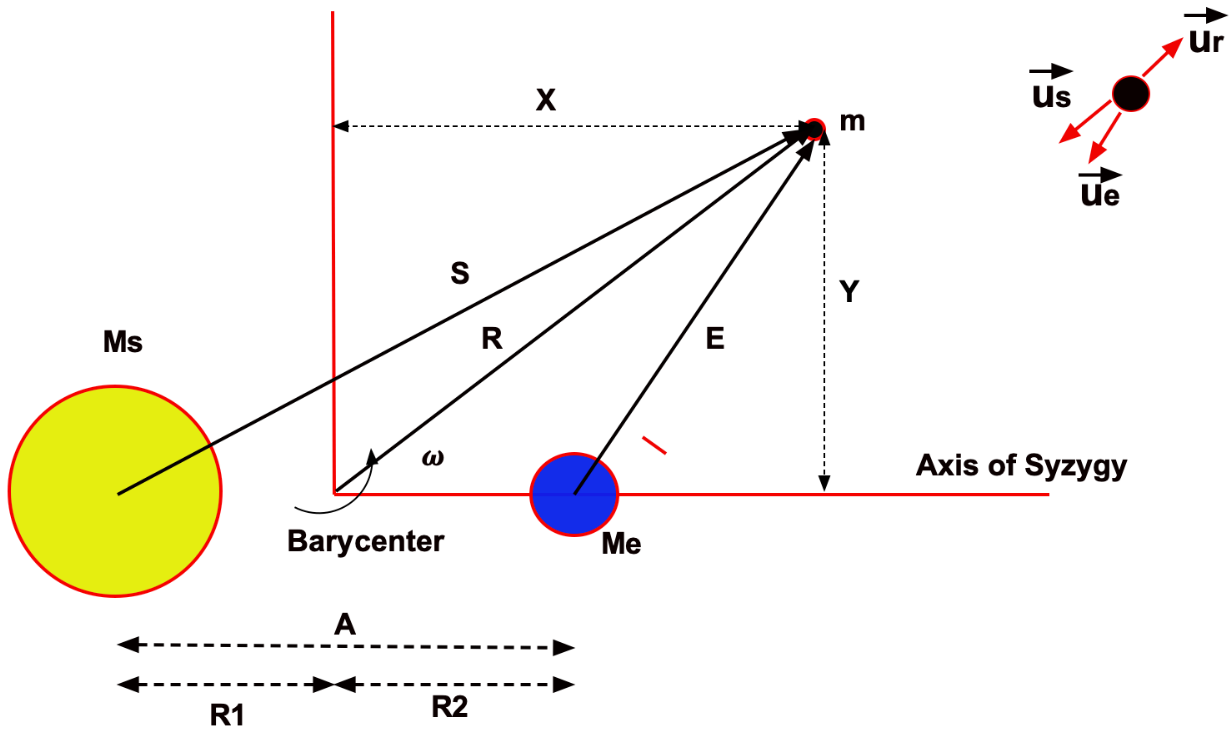


Fig 2 The Three Body Problem

Figure 2 depicts the earth, sun, satellite system. Whereas many believe the earth rotates around the sun, that is not true. The system sun-earth rotates around its center of mass called the barycenter. The red axis in Figure 5 is centered at the barycenter, and rotates at an angular velocity ω in the inertial frame. The coordinates x and y give the position of the satellite within the rotating frame. R1 and R2 are the distances from the mass centers to the barycenter, while A is the separation between the earth and sun. The center of mass is given by the following equation. Here R_1 and R_2 are the distances from the sun and the earth to the center of mass called the barycenter.

$$M_s R_1 = M_e R_2$$

(1) $$R_2 = A - R_1$$

Divide both sides by A and define $\mu = R_1/A$

$R_2/A = 1 - \mu$

Let us define the dimensionless parameter μ, we then obtain the following.

(2) $$\frac{M_e}{M_s} = \frac{R_1/A}{R_2/A} = \frac{\mu}{1-\mu}$$

The gravitational forces are given in equation (3). Here $\overrightarrow{u_s}$ and $\overrightarrow{u_e}$ are unit vectors pointing toward the sun and earth, respectively.

$$\overrightarrow{F_s} = \frac{mM_sG}{{D_s}^2}\overrightarrow{u_s} = \frac{mM_sG}{A^2}\frac{1}{{d_s}^2}\overrightarrow{u_s} \qquad \text{where } d_s = \frac{D_s}{A}$$

(3) $$\overrightarrow{F_e} = \frac{mM_s\mu G}{(1-\mu){D_e}^2}\overrightarrow{u_e} = \frac{mM_sG}{A^2}\frac{\mu}{1-\mu}\frac{1}{{d_e}^2}\overrightarrow{u_e} \qquad \text{where } d_e = \frac{D_e}{A}$$

$$\overrightarrow{F_r} = mR\omega^2\overrightarrow{u_r} = Am\omega^2r^2\overrightarrow{u_r} \qquad \text{where } r = \frac{R}{A}$$

The centrifugal force arises from the centripetal acceleration, and the application of D'Alembert's principle. The centrifugal force acts along the radius from the barycenter to the satellite, and is pointed away from the barycenter. Here the unit vector $\overrightarrow{u_r}$ is directed away from the barycenter. Note that the direction of the unit vectors indicates that the gravitational forces are opposed by the centrifugal force.

We can find the angular frequency of earth's orbit around the barycenter.

$$\frac{GM_sM_e}{A^2} = M_eR_22\omega^2$$

$$\omega^2 = \frac{M_s G}{A^2 R_2}$$

$$\overrightarrow{F_r} = mR\frac{M_s G}{A^2 R_2}\overrightarrow{u_r}$$ but $R_2 = A(1-\mu)$

$$\overrightarrow{F_r} = mR\frac{M_s G}{A^3(1-\mu)}\overrightarrow{u_r} = \frac{mM_s G}{A^2}\frac{r}{1-\mu}\overrightarrow{u_r}$$

The parameters in this equation are given in the following figure.

Physical Parameters

$$M_e = 5.9722 \times 10^{24} kg$$

$$M_s = 1.989 \times 10^{30} kg$$

$$A = 149.597870 \times 10^6 km$$

$$G = 6.67430 \times 10^{-11} \frac{km}{kgs^2}$$

Computed Parameters

$$\rho = 3.003 \times 10^{-6}$$

$$R_1 = .00449183 \times 10^6 km$$

$$R_2 = 149.597 \times 10^6 km$$

Fig. 3 sun/earth constants

Rather than dealing with the large distances which appear in these equations, we will define all distances divided by A, the distance between the earth and the sun. Thus, the non dimensional distance between the earth and the Webb is given by $d_s = \dfrac{D_s}{A}$. We will use upper case letters to represent the actual distance, and lower case letters to represent non dimensional distances.

If we define $K = \dfrac{mM_sG}{A^2}$

$$\overrightarrow{F_s} = \frac{mM_sG}{D_s^{\,2}}\overrightarrow{u_s} = K\frac{1}{d_s^{\,2}}\overrightarrow{u_s} \qquad \text{where } d_s^{\,2} = (x+\mu)^2 + y^2$$

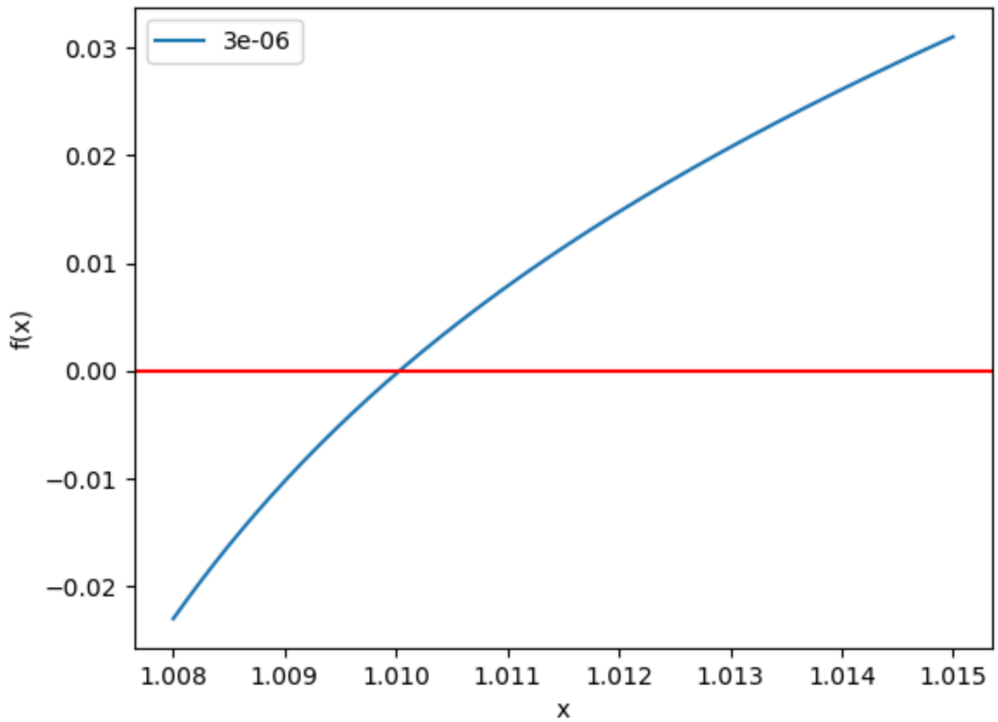
$$\overrightarrow{F_e} = \frac{mM_s\mu G}{(1-\mu)E^2}\overrightarrow{u_e} = K\frac{\mu}{1-\mu}\frac{1}{e^2}\overrightarrow{u_e} \qquad \text{where } d_e^{\,2} = (x-(1-\mu))^2 + y^2$$

$$\overrightarrow{F_r} = mR\frac{M_sG}{A^3(1-\mu)}\overrightarrow{u_r} = K\frac{r}{1-\mu}\overrightarrow{u_r} \qquad \text{where } r^2 = x^2 + y^2$$

For L2, the unit vectors for the gravitational forces are all directed along the axis of Syzygy toward the left, while the centrifugal force is directed toward the right. The sum of forces at L2=0.

$$K\left[\frac{1}{(x+\mu)^2} - \frac{\mu}{1-\mu}\frac{1}{(1-\mu-x)^2} + \frac{x}{1-\mu}\right] = 0$$

L2 was

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L2= [1.01003023]
potential at L2= [6.15063556e-13]
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Fig. 4 Force on Axis of Syzygy near L2

determined to be 1.01003023 in scaled units. Multiplying this by A we obtain the distance of L2 from the barycenter is given as

$$L2 = 151.0983 \times 10^6 km$$

Potential Energy

The two gravitational forces give rise to potential energy. Like the forces due to gravity, the centrifugal force depends on position (radial distance

from the barycenter), and not velocity. For this reason, we can write a potential function for this force, as well. The change in potential energy U_{OP} between point O and point P is defined as the work required to move the satellite from the barycenter O to the orbital position P. There are three forces influencing this work—two gravitational and one centrifugal. We can define the total work as the sum of the work done against each of these three forces.

$$W = W_{OB} + W_{BP} + W_{OP} + W_{OA} + W_{AP}$$

$$U_P = U_O + W$$

In work done in traversing the two arcs W_{OA} and W_{OB} under the influence of the gravitational fields is zero. In both cases, the motion is perpendicular to the force. We are then left with three components of work to deal with. We can write

$$W_{BP} = K\int_{\mu}^{d_s} \frac{1}{\sigma^2} d\sigma = -\frac{1}{\sigma}\Big|_{\mu}^{d_s} = -K\Big[\frac{1}{d_s} + \frac{1}{\mu}\Big]$$

$$W_{AP} = K\frac{\mu}{1-\mu}\int_{1-\mu}^{d_e} \frac{1}{\sigma^2} d\sigma = -K\frac{\mu}{1-\mu}\frac{1}{\sigma}\Big|_{1-\mu}^{d_e} = -K\frac{\mu}{1-\mu}\Big[\frac{1}{d_e} - \frac{1}{1-\mu}\Big]$$

The work don in moving from O to P along r is negative. That is, the centrifugal force is pulling a mass away from O.

$$W_{OP} = -K\frac{1}{1-\mu}\int_0^r \sigma d\sigma = -K\frac{1/2}{1-\mu}r^2$$

The constant $K = \dfrac{mM_sG}{A^2}$ contains the mass m located at point P. We can generalize the equations and define the work per unit mass by dividing by m. An equation for the total potential per unit mass is then given by

$$U = \frac{M_sG}{A^2}\Big[-\frac{1}{d_s} - \frac{\mu}{1-\mu}\frac{1}{d_e} - \frac{1/2}{1-\mu}r^2\Big] + C$$

If we multiply all terms by $(1-\mu)$ and drop the constant term, we obtain a scaled potential energy u.

$$u = -\frac{1-\mu}{d_s} - \frac{\mu}{d_e} - \frac{1}{2}r^2$$

(4)

$$r^2 = x^2 + y^2$$

$$d_s^{\,2} = (x+\mu)^2 + y^2$$

$$d_e^{\,2} = (x-1+\mu)^2 + y^2$$

The potential energy contours near L2 for the earth/sun system are shown in fig. X3a; the surface map of the potential energy follows in X3b. It is clear that although at L2 the force on a particle is zero, it is not a stable point. Rather, it is a saddle. A small perturbation along the x axis will either send the mass ever accelerating into space, or toward the gravitational well of earth. It should be noted that the analysis we have done applies to a stationary mass. In a rotating frame, there is an additional fictitious force which is due to the corollas acceleration. We will discuss this later when we describe the orbital dynamics.

The second point to be made is that there are a set of five Lagrange points associated with the earth/sun system. Three of them are on the axis of sygyzy. L1 is between the earth and the sun. Two additional points L3 and L4 are located where the orbital path of the earth intersects radial lines drawn from earth's center and inclined at xxxx. Finally, the Lagrange points exist for each planet orbiting a star. More detail is found in Appendix A.

The net gravitational and centrifugal force is given by the gradient of this potential function. This is shown at the right by the green arrows superimposed on the contours. We cannot simply write Newton's second law using these forces. Newton's law describes motion in an inertial frame.

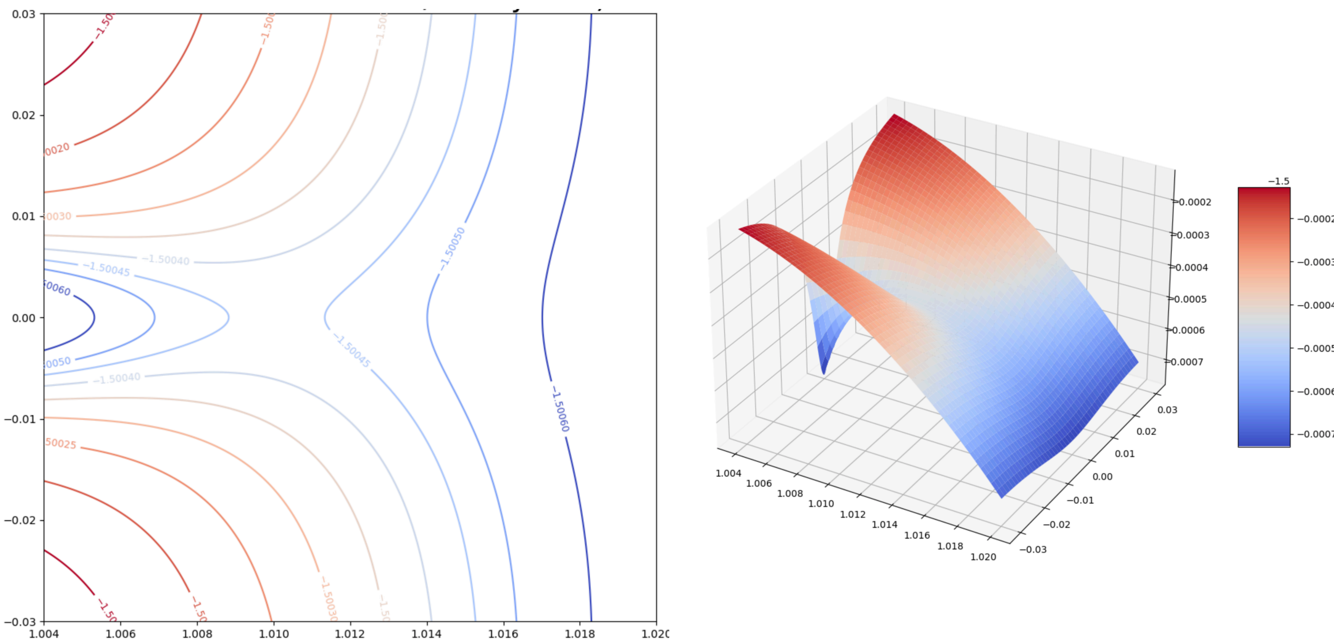

Fig. 6 Contour and Surface Plot of the Potential Function

These notes are not complete