Equation for Seasonal Variation of Daylight

Since the earth's rotational axis is inclined at an angle from the normal of the orbital plane, we experience a periodic change in the length of the day during the annual orbit of the earth about the sun. The question we will explore is what is the nature of this periodic fluctuation? Is it purely sinusoidal, or is a more complex function of orbital position.

Orbital Geometry

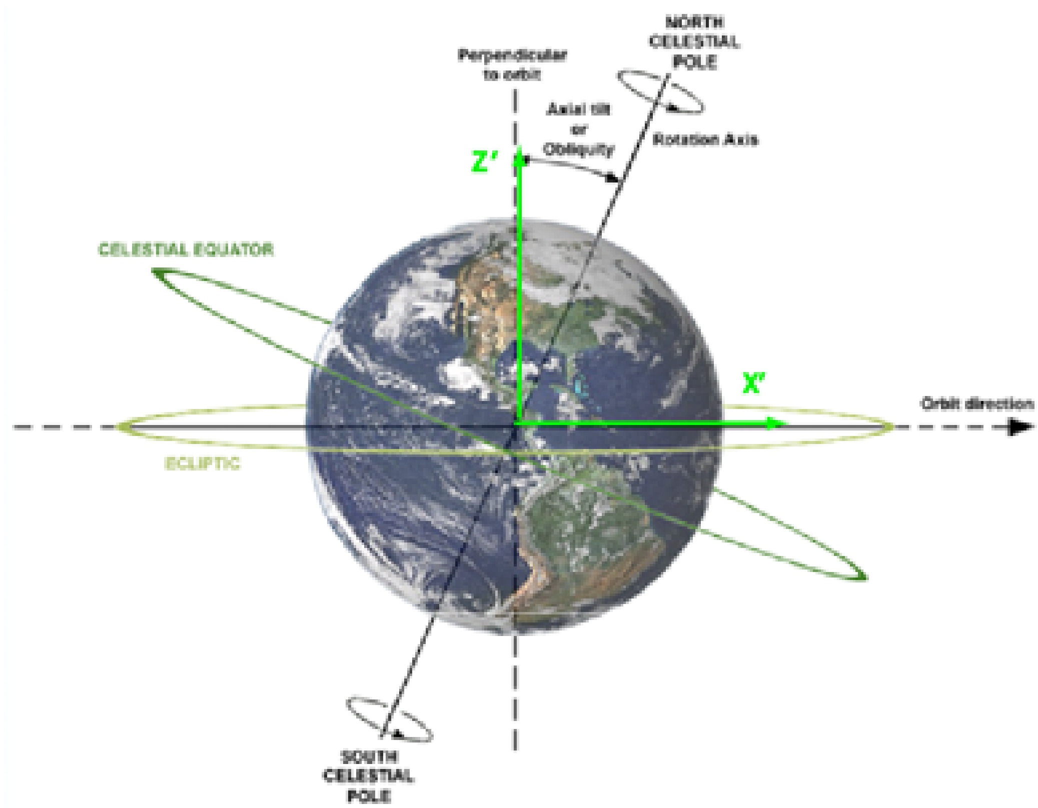

Figure 1: Earth-Based Coordinate Frame

We assume that the earth is perfectly spherical and that the earth's orbit is a perfect circle. We know that this is not true, but it is a reasonable first approximation. Let us define a plane that contains the rotational axis of earth and the perpendicular to the earth's orbit. We will define a coordinate system (X',Y',Z') embedded in the center of the earth; the (X',Y') plane lies in the plane of the earth's orbital path around the sun. The polar axis of rotation of the earth is contained within the (X',Z') plane. In the figure above, the Y' axis is normal and facing inward toward the back of the page.

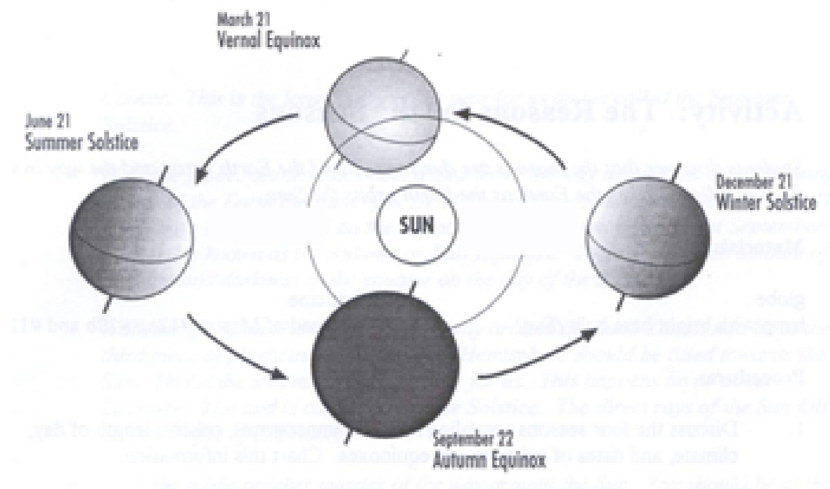

Figure 2: Seasonal Markers in Earth's Orbit

The (X'Y',Z') coordinate frame has a fixed orientation in space as the earth orbits the sun[1]. It translates, but does not rotate. The location of the pole is shown by the small red dot in the figure below.

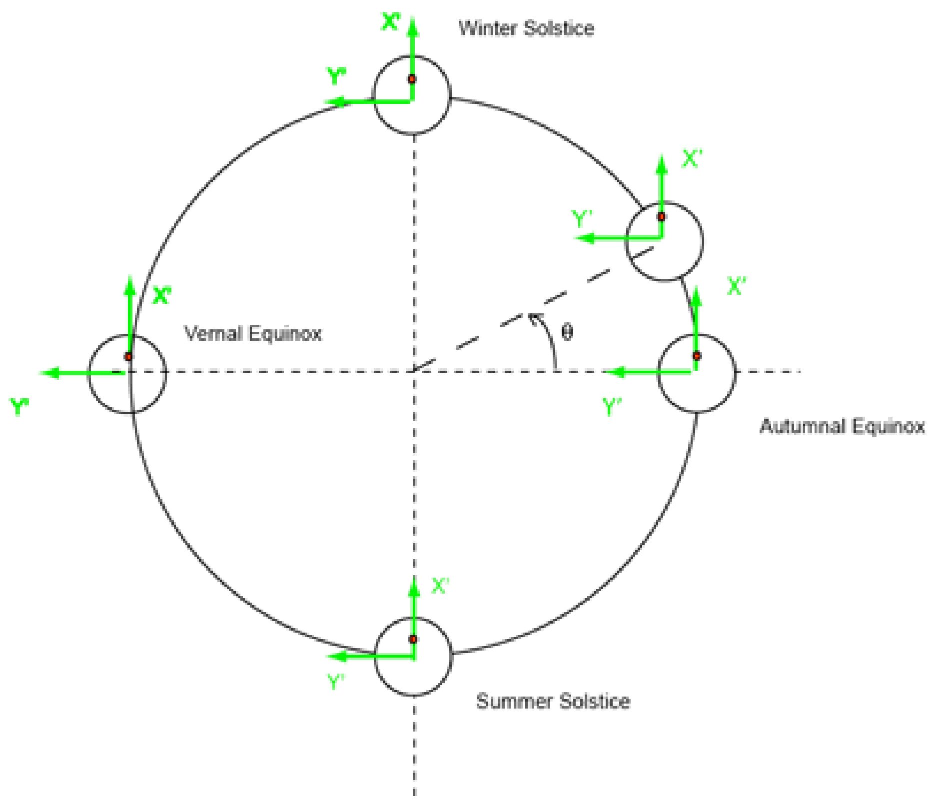

Figure 3: Orientation of the (X'Y') Coordinate Frame

In Figure 3 the Z' axis faces upward from the orbital plane, out from the page.

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[1] The normal to the orbital plane remains fixed and the polar axis maintains a fixed orientation in space (pointing toward Polaris) due to gyroscopic forces.

We now introduce a new coordinate system (X,Y,Z) related to (X',Y',Z'). In this new coordinate system the origins as well as the Z and Z' axes are identical; the (X,Y) and (X',Y') axes are coplanar. The Y axis of the new coordinate system always points toward the sun. In making this definition we will be able to define the illumination disc of the incident solar radiation on the earth as lying in the (X,Z) plane with Y=0. At the autumnal Equinox both coordinate frames are oriented identically.

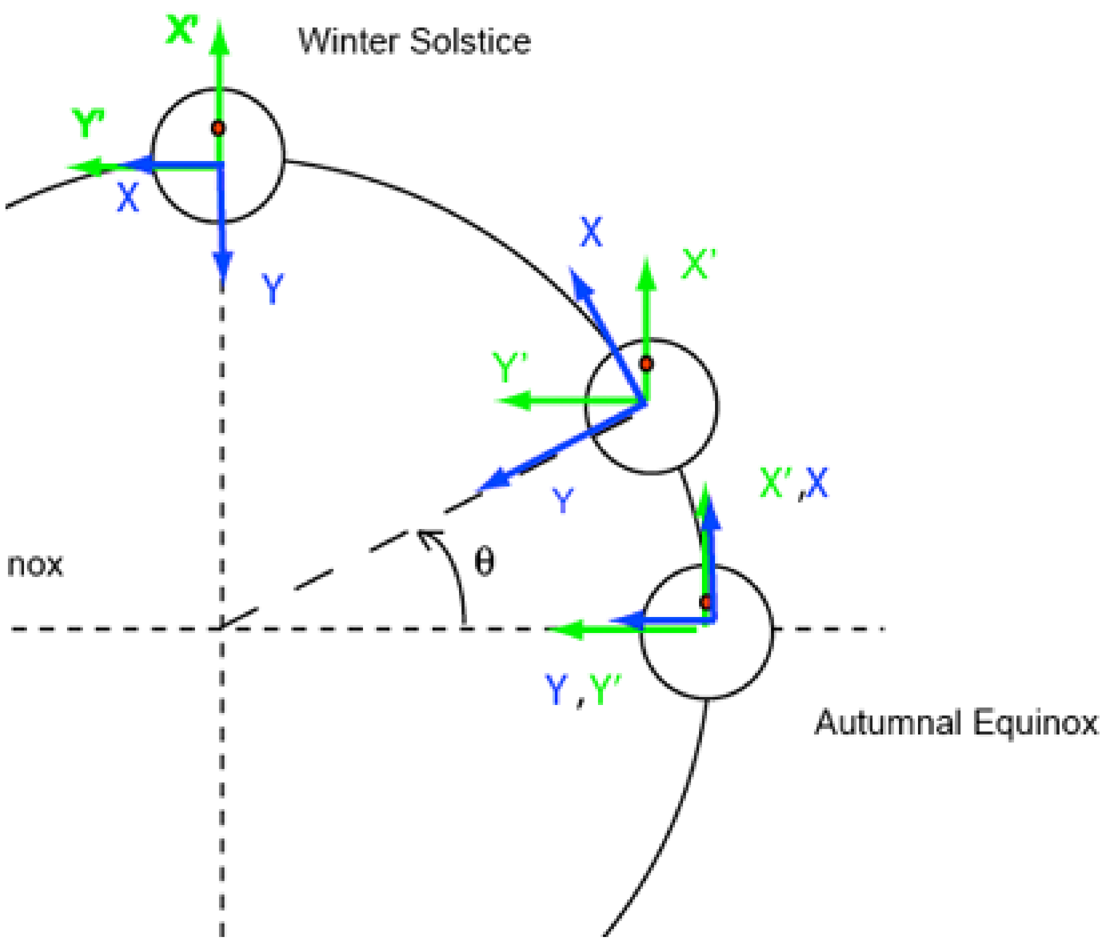


Figure 4: The Plane of the Solar Illumination Disk

A vector $\vec{V}$ can be represented in both the primed and unprimed coordinate systems. $\left[\vec{I} \quad \vec{J} \quad \vec{K}\right]^T$ and $\left[\vec{I}' \quad \vec{J}' \quad \vec{K}'\right]^T$ are the unit vectors in the unprimed and primed coordinate systems respectively.

(1) $$\vec{v} = X\vec{I} + Y\vec{J} + Z\vec{K} = X'\vec{I}' + Y'\vec{J}' + Z'\vec{K}'$$

Forming the scalar product of equation (1) with each of the unit vectors in the unprimed coordinate system we obtain equation (2)

(2) $$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \vec{I}' \cdot \vec{I} & \vec{I}' \cdot \vec{J} & \vec{I}' \cdot \vec{K} \\ \vec{J}' \cdot \vec{I} & \vec{J}' \cdot \vec{J} & \vec{J}' \cdot \vec{K} \\ \vec{K}' \cdot \vec{I} & \vec{K}' \cdot \vec{J} & \vec{K}' \cdot \vec{K} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

(3) $$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We next introduce a transformation by *rotating about* the Y' axis to form the frame (x,y,z). The rotational angle[2] is β . This will align the z axis with the axis of rotation of the earth and the x and y axes will intersect earth's equator.

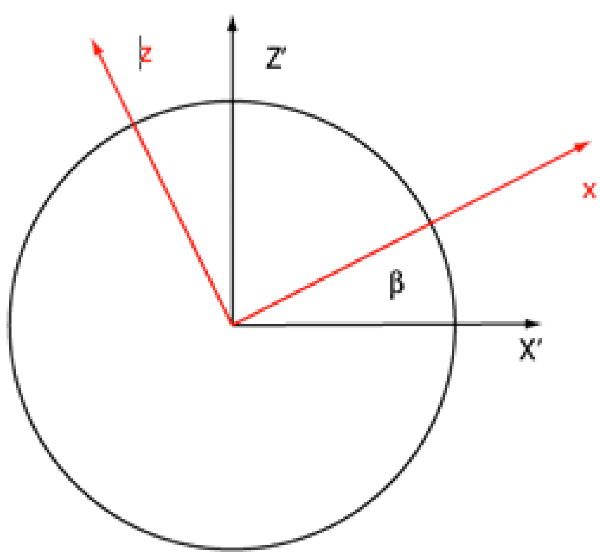


Figure 5: Rotating the axis to align with the polar axis

Applying the same coordinate transformation approach to the development of (2) we obtain equation 4.

[2] All axis systems are right-hand systems, that is the positive direction of the z axis (for example is determined by applying the right hand rule (RHR)

(4) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

The earth's pole is inclined at an angle of about -23.5 degrees in this coordinate system. (Note positive angles are measured as aligning with the positive axis according to the right hand rule). Let us use the symbol γ_o to represent the tilt angle of 23.5 degrees. As the earth orbits the sun, $\theta = 2\pi t/T$ where T is the orbital period of the earth about the sun[3]. The net coordinate transformation is given by the following. The transformation from (X,Y,Z) to the (x,y.z) is given by the product of equation (4) and (3).

$$\begin{bmatrix} \cos\gamma_o & 0 & \sin\gamma_o \\ 0 & 1 & 0 \\ -\sin\gamma_o & 0 & \cos\gamma_o \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\gamma_o & -\sin\theta\cos\gamma_o & \sin\gamma_o \\ \sin\theta & \cos\theta & 0 \\ -\sin\gamma_o\cos\theta & \sin\theta\sin\gamma_o & \cos\gamma_o \end{bmatrix}$$

(5) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\gamma_o & -\sin\theta\cos\gamma_o & \sin\gamma_o \\ \sin\theta & \cos\theta & 0 \\ -\sin\gamma_o\cos\theta & \sin\theta\sin\gamma_o & \cos\gamma_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We are now in a position to express the equation of the solar illumination plane in the earth-based coordinates (x,y,z). Since the Y axis always points toward the sun, then the illumination disk is given by Y=0, for all X,Y. We know that the following points lie in this plane: (0,0,0) ; (1,0,0) and (0,0 ,1). Thus, three points in the illumination plane are

(6.1) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\theta\cos\gamma_o & -\sin\theta\cos\gamma_o & \sin\gamma_o \\ \sin\theta & \cos\theta & 0 \\ -\sin\gamma_o\cos\theta & \sin\theta\sin\gamma_o & \cos\gamma_o \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

[3] We measure θ along the orbital path with θ=0 occurring at the autumnal equinox.

(6.2) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta\cos\gamma_o & -\sin\theta\cos\gamma_o & \sin\gamma_o \\ \sin\theta & \cos\theta & 0 \\ -\sin\gamma_o\cos\theta & \sin\theta\sin\gamma_o & \cos\gamma_o \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\gamma_o \\ \sin\theta \\ -\sin\gamma_o\cos\theta \end{bmatrix}$$

(6.3) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_3 = \begin{bmatrix} \cos\theta\cos\gamma_o & -\sin\theta\cos\gamma_o & \sin\gamma_o \\ \sin\theta & \cos\theta & 0 \\ -\sin\gamma_o\cos\theta & \sin\theta\sin\gamma_o & \cos\gamma_o \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\gamma_o \\ 0 \\ \cos\gamma_o \end{bmatrix}$$

The equation of the solar illumination plane in (x,y,z) coordinates is given by

(7) x = by +cz + d

From (6.1) d must be zero.

Using (6.2) and (6.3)

(8) $$\begin{bmatrix} \cos\theta\cos\gamma_o \\ \sin\gamma_o \end{bmatrix} = \begin{bmatrix} \sin\theta & -\sin\gamma_o\cos\theta \\ 0 & \cos\gamma_o \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}$$

Thus,

(9) $$c = \frac{\sin\gamma_o}{\cos\gamma_o}$$
$$b = \frac{\cos\theta}{\sin\theta\cos\gamma_o}$$

The constant c is given by the tangent of the angle of inclination of the polar axis, which is 23.5 degrees. Thus,

$$c = \frac{\sin\gamma_o}{\cos\gamma_o} = \tan(23.5^\circ) = 0.435$$

$$b = \frac{\cos\theta}{\sin\theta\cos\gamma_o} = \frac{1.09}{\tan\theta}$$

Geometry on the Earth's Surface

The normal coordinate frame for earth-based observations is the coordinate system centered at the core with the z axis lying along the axis of rotation, namely (x,y,z).

(10) $$x^2 + y^2 + z^2 = R^2$$

where R is the radius of the earth. At the latitude ϕ, points contained on the circle of constant latitude is given by the equation of a circle. We can also eliminate z from equation (7) describing the illumination plane.

(11) $$\begin{aligned} x^2 + y^2 &= R^2 \cos^2 \phi \\ x &= by + cR\sin\phi \end{aligned}$$

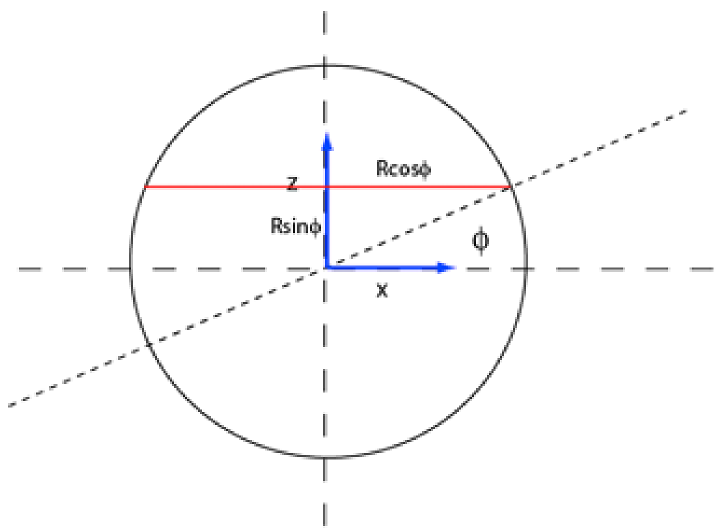


Figure 6: Circles of Equal Latitude

As the Earth orbits the sun (see Figure 7) from autumnal equinox (right) to winter solstice (top) to vernal equinox (left) the solar illumination disc intersects a circle of constant latitude; this is shown by the green line. The intersection points are shown by red dots. At each position of the earth around the orbit, the camera view is directly above the north pole (not normal to orbital path)[4]. At the equinox the intersection passes through the poles and the length of day and night are equal. The largest distance the illumination disk moves from the pole occurs at

[4] While the geometry is distorted in this illustrative figure, it is preserved in the mathematics we have developed.

the winter solstice; Here the illumination plane is tangent to the Arctic Circle, and north of this point the sun does not rise.

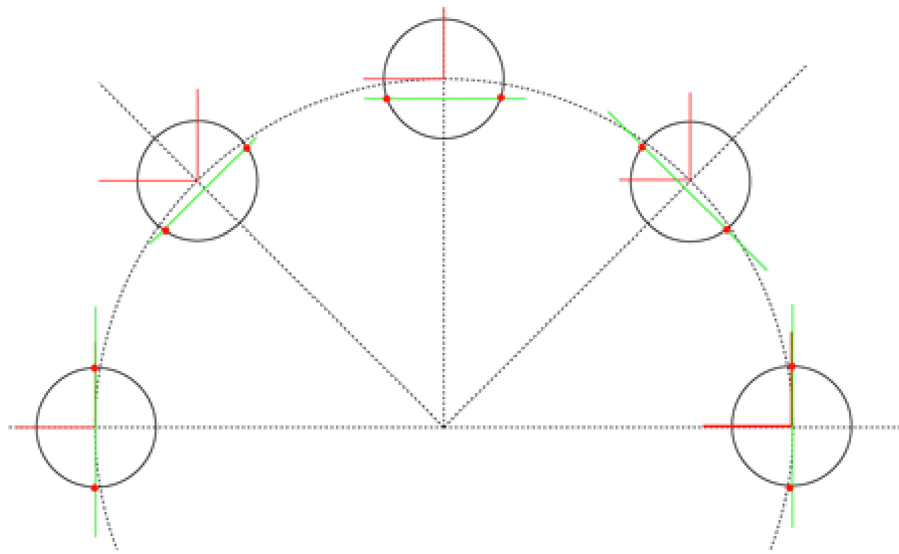


Figure 7: Intersection of the Illumination Disk with a Circle of Equal Latitude.

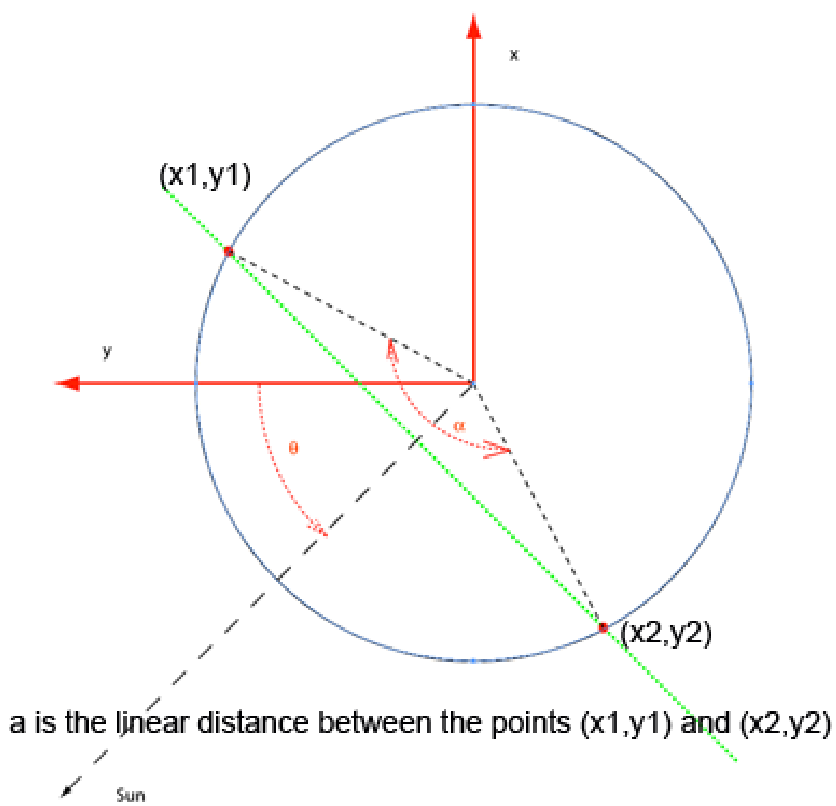


Figure 8: Detail of Intersection

In Figure 8 we take a close-up view of the earth half-way between the the autumnal equinox and the winter solstice. Dividing the angle α by 2π gives the fraction of daylight time. At the equinox $\alpha=\pi$. During the winter months $\alpha<\pi$, while during the summer months $\alpha>\pi$.

By the law of cosines we can relate the angle α to the intersection points (x1,y1) and (x2,y2).

$$\text{(12)} \quad \cos\alpha = \frac{2R^2\cos^2\phi - a^2}{2R^2\cos^2\phi} = 1 - \frac{(x_1-x_2)^2 + (y_1-y_2)^2}{2R^2\cos^2\phi}$$

The details of the next step are found at the end of this paper in the section Detail A. From equation (A.9)

$$\text{(13)} \quad \cos\alpha = \frac{2R^2\cos^2\phi - a^2}{2R^2\cos^2\phi} = 1 - 2\left[1 - \frac{c^2}{\left(1+b^2\right)}\tan^2\phi\right]$$

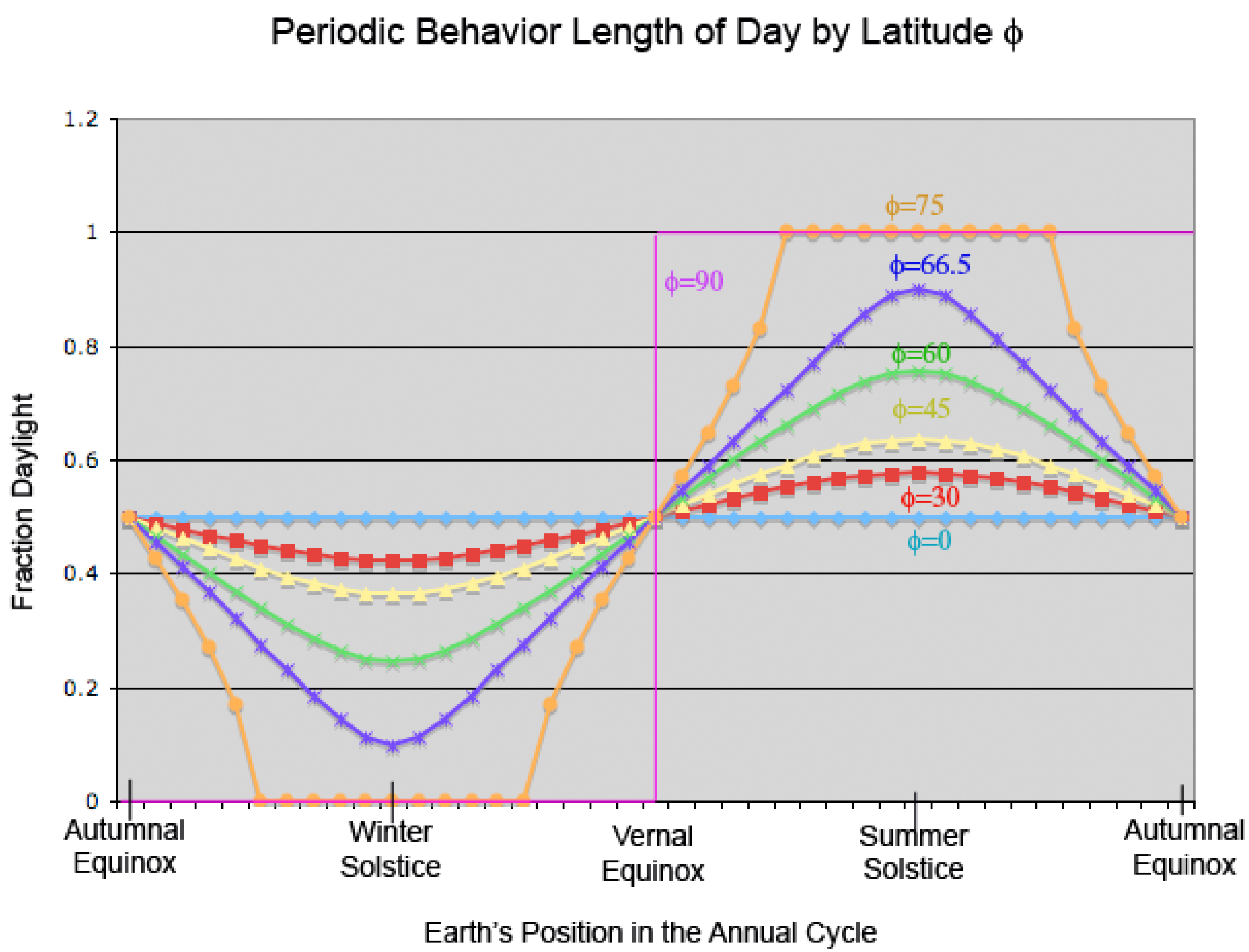
but since

$$c = -\frac{\sin\gamma}{\cos\gamma}$$

$$b = \frac{\cos\theta}{\sin\theta\cos\gamma}$$

$$\text{(14)} \quad \cos\alpha = -1 + \frac{2\sin^2\theta\sin^2\gamma_0}{\left(\cos^2\gamma_0\sin^2\theta + \cos^2\theta\right)}\tan^2\phi$$

Figure 9 displays the periodic behavior of the length of the day by latitude. At the equator the length of the day and night are invariant with the passing of seasons. At the poles, the cycle is a square wave. On the summer side of the equinox, the pole has 24 hours of daylight, while in the winter it is plunged into darkness for 24 hours a day. For temperate latitudes, the waveform appears to be very close to a sine wave. Above the Artic Circle, it clearly is not.


Periodic Behavior Length of Day by Latitude ϕ
Fraction Daylight
1.2
1
0.8
0.6
0.4
0.2
0
ϕ=75
ϕ=66.5
ϕ=90
ϕ=60
ϕ=45
ϕ=30
ϕ=0
Autumnal Equinox
Winter Solstice
Vernal Equinox
Summer Solstice
Autumnal Equinox
Earth's Position in the Annual Cycle

At the latitude of San Francisco (37.775N) we can fit a sine function to the data so that the maximum length of day predicted by the theoretical model is the same as that predicted by the fitted sine function. The fitted model always predicts longer days except at the equinox and the solstice at which it is right on. However, it is remarkable how close the fitted curve matches the theoretical. The largest deviation for San Francisco is 0.004. Of course the fitted curve does not predict the length of the day; this is data obtained from the theoretical model.

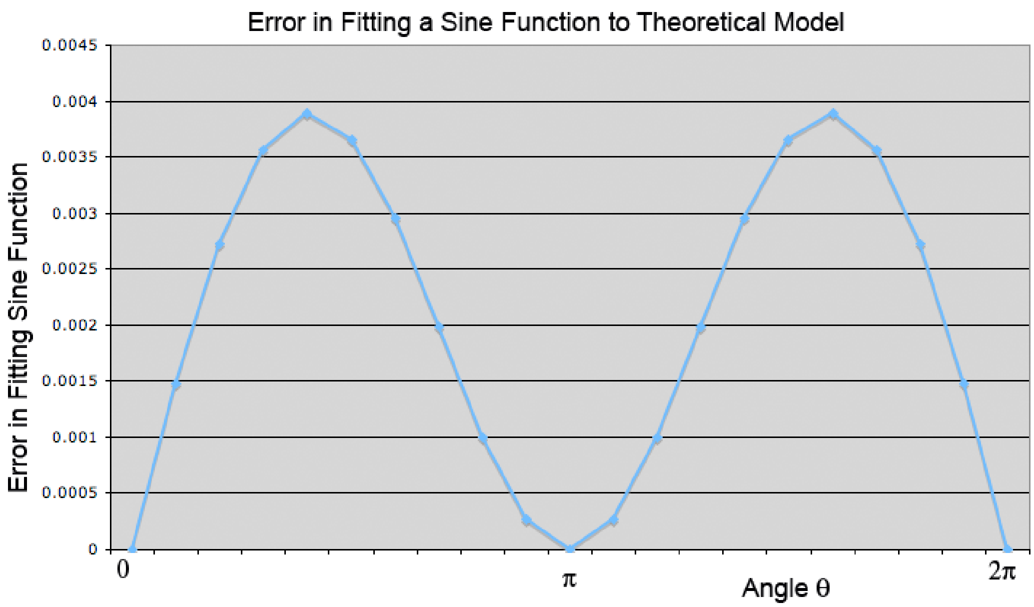


Figure 10: Error in Fitting Sine Function to Theoretical Model

Detail A Supporting Equation (13)

What are the values of x,y that satisfy

(A.1)
$$x^2 + y^2 = R^2 \cos^2 \phi$$
$$x = by + cR\sin\phi$$

It appears that we could eliminate x and solve the resulting quadratic equation for y. There is a computational problem in doing this. At the equinoxes θ=0, and therefore b=∞. The resulting indeterminate form poses computational problems. If, however, we eliminate y from the equation of the circle, we can avoid these problems.

(A.2)
$$y = \frac{1}{b}x - \frac{cR}{b}\sin\phi$$

Now when θ=0, y=0 which is what we would expect by examining Figure 8. Substituting (A.2) this into the equation of the circle we obtain the following.

(A.3)
$$x^2 + \frac{1}{b^2}\left(x - cR\sin\phi\right)^2 = R^2\cos^2\phi$$
$$\left(1 + \frac{1}{b^2}\right)x^2 + \frac{2cR\sin\phi}{b^2}x + R^2\left(\frac{c^2}{b^2}\sin^2\phi - \cos^2\phi\right) = 0$$

Let us examine the case when we are at the equinox. Here θ=0 and b=∞. Then,

(A.4)
$$y = 0$$
$$x = \pm R\cos\phi$$

The length of the cord between intersection points of the illumination disc and the circle of constant latitude is $2R\cos\phi$. From equation (12) we obtain the fact that for all ϕ

(A.5)
$$\cos\alpha = \frac{2R^2\cos^2\phi - \left(2R\cos\phi\right)^2}{2R^2\cos^2\phi} = -1$$
$$\alpha = \pi$$

For values of $\theta > 0$ Multiply equation (A.4) by b^2.

$$(b^2+1)x^2 + 2cR\sin\phi x + R^2\left(c^2\sin^2\phi - b^2\cos^2\phi\right) \tag{A.6}$$

The roots of equation (A.6) can be found.

$$x_{1,2} = \frac{-cR\sin\phi \pm \sqrt{c^2R^2\sin^2\phi - R^2(b^2+1)\left(c^2\sin^2\phi - b^2\cos^2\phi\right)}}{(b^2+1)}$$

$$x_{1,2} = \frac{-cR\sin\phi \pm Rb\sqrt{(1+b^2)\cos^2\phi - c^2\sin^2\phi}}{(1+b^2)}$$

Computing the difference between x_1 and x_2 we find

$$x_1 - x_2 = \frac{2Rb\sqrt{(1+b^2)\cos^2\phi - c^2\sin^2\phi}}{(1+b^2)}$$

$$(x_1 - x_2)^2 = \frac{4R^2b^2\left[(1+b^2)\cos^2\phi - c^2\sin^2\phi\right]}{(1+b^2)^2} \tag{A.7}$$

We can now find the difference in the y coordinates.

$$y_1 - y_2 = \frac{1}{b}(x_1 - x_2)$$

$$(y_1 - y_2)^2 = \frac{4R^2\left[(1+b^2)\cos^2\phi - c^2\sin^2\phi\right]}{(1+b^2)^2} \tag{A.8}$$

Finally, we can compute the square of the length of the cord between intersection points in Figure 8. Here we return to make this substitution in equation (12).

$$(A.9)\qquad \left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 = 4R^2\left[\cos^2\phi - \frac{c^2}{\left(1+b^2\right)}\sin^2\phi\right]$$